Section 2.1: Electrical systems: Basics review

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Introduction

Objective
The aim of this section is to recall the basic governing laws of electrical circuits.

Prior knowledge needed
Knowledge of basic electrical components (resistor, capacitor, inductor), voltage and current
Knowledge of the complex notation in the harmonic domain (see section 1.2).
We recommend that you test your current knowledge. If you do not succeed, you may need to review the basic notions (see section 1.2), or the required notions.

**Exercice 1 : Test your knowledge**

**Question 1**

*What is the unit of the voltage $v$?*

- Volt
- Ampère
- Coulomb
- Watt
- Faraday
- Ohm
- Joule

**Question 2**

*What is the unit of current $i$?*
### Test your prior knowledge

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<th>Option</th>
<th>Description</th>
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<td>Volt</td>
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### Question 3

*What is the unit of electrical charge $q$?*

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<th>Option</th>
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<td>Volt</td>
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### Question 4
When a current flows between two terminals of a conductor:

- There must be no voltage drop between these points
- There must be a voltage drop between these points
- There is a circulation of electrical charges

**Question 5**

What is the relation between the electrical charge $q(t)$ and the current $i(t)$?

- $i(t) = q(t)$
- $i(t) = \frac{dq(t)}{dt}$
- $q(t) = \frac{di(t)}{dt}$
- $i(t) = \frac{d^2q(t)}{dt^2}$
- $q(t) = \frac{d^2i(t)}{dt^2}$
- $i(t) = \int q(t)dt$
- $q(t) = \int i(t)dt$
A. Two terminal networks

1. The notion of two terminal networks

A two terminal network is an electrical component which has two terminals. Light bulbs, batteries, switches, resistors and motors are examples. We distinguish between two types of two terminal networks:

- Generators: active two terminal networks,
- Receivers: passive two terminal networks.

2. Conventions for the orientation of voltage drops and current

The conventional orientation of the voltage and current is illustrated in the following image:

- for generators, the voltage and the current both go in the same direction (left part of the figure),
B. Quadripoles (two port networks)

1. Notions of quadripoles (two port networks, or four terminal networks)

A quadripole is a system with two inputs, each one has two poles. A quadripole allows an energy transfer between two dipoles connected to either input. The description of a dipole requires the use of four physical quantities:
- The voltage drop across each input,
- the current entering each input.

**Example**

The single phase electrical transformer is a quadripole.
2. Orientation conventions

There are two ways to represent a quadripole:

- symmetrical convention: all the currents enter the quadripole. It is therefore seen as a receiver from each side (see reference 9)
- Asymmetrical convention: the quadripole is seen as a system with an output and an input. The current enters into the left input, and exits out of the right input. Therefore, the receiver convention is used on the left part, and the generator convention used on the right (see ref 6).

3. Relations between conventions

The two conventions presented above affect the connection laws between quadripoles.
Symmetrical convention

Under this convention, the sum of all currents in the node between the quadripoles is null \( i_{s1} + i_{s2} = 0 \) and the voltage drops are equal \( U_{s1} = U_{e2} \). The transfer matrix between the two quadripoles is therefore written:

\[
\begin{pmatrix}
    1 & 0 \\
    0 & -1
\end{pmatrix}
\begin{pmatrix}
    U_{e2} \\
    i_{e2}
\end{pmatrix}
\]

Asymmetrical convention

Under this convention, the difference between the currents in the node between the two quadripoles is null \( i_{s1} - i_{s2} = 0 \) and the voltage drops are equal \( U_{s1} = U_{e2} \). The transfer matrix between the two quadripoles is therefore written:

\[
\begin{pmatrix}
    1 & 0 \\
    0 & 1
\end{pmatrix}
\begin{pmatrix}
    U_{e2} \\
    i_{e2}
\end{pmatrix}
\]

4. The point of conventions

Symmetrical conventions

It is used for the analysis of the energy conversion between different domains (electric ↔ mechanic and mechanic ↔ acoustic).

This convention shows that the total power injected into the quadripole is null.

Asymmetrical convention

For mechanical or acoustical systems, for which a source is used before the quadripole, and a receiver after, the asymmetrical convention is preferred to show the continuity of speed or flow.

Example: Acoustic transmission line

The schematic on the left shows a real acoustic system, and the one on the right the equivalent electrical circuit at low frequency. It uses three quadripoles under the asymmetrical convention.
C. Complex notation in the harmonic domain

Reminders

With the hypothesis that the current and voltage are time dependent following a sine law, these variables are written

- for the voltage: \( u(t) = u_m \cos(\omega t + \phi_u) \),
- for the current: \( i(t) = i_m \cos(\omega t + \phi_i) \).

Using the complex notation (see Section 2.1, reminders: basic notions) these become

- for the voltage: \( u(t) = u_m e^{j(\omega t + \phi_u)} \),
- for the current: \( i(t) = i_m e^{j(\omega t + \phi_i)} \).

The respective time derivatives are written

- for the voltage: \( \frac{du(t)}{dt} = j\omega u(t) \),
- for the current: \( \frac{di(t)}{dt} = j\omega i(t) \).

The respective integrals of the voltage and current can be written

- for the voltage: \( \int u(t) dt = \frac{u(t)}{j\omega} \),
- for the current: \( \int i(t) dt = \frac{i(t)}{j\omega} \).
Ohms law

In the case of an electrical resistance, the relation between the macroscopic voltage $u(t)$ and current $i(t)$ is Ohms law, given by:

$$u(t) = Ri(t),$$

where $R$ is the resistance in Ω (Ohms).

**Complément : Additional resources**

In reality, the current that passes through the resistance increases its temperature by the Joule effect. This in turn then modifies the value of the resistance. The resistance is therefore time dependant, and should be written $R(t)$.

The power dissipated by the resistance is written

$$P_e(t) = u(t)i(t) = Ri(t)^2 = \frac{u(t)^2}{R}$$
A. Contents of an RLC circuit

The RLC circuit is composed of a resistor $R$, an inductor (a self) $L$ and a capacitor $C$. These components can be connected in series or in parallel (see the figure below).
B. Time domain

The relations between voltage and current for the classic dipoles (resistor, inductor, capacitor), in the time domain, are written:

- $U_R(t) = Ri(t)$, where $i(t)$ is the current flowing through the resistor and $U_R(t)$ the voltage drop across the resistor.
- $u_L(t) = L \frac{di(t)}{dt}$, where $i(t)$ is the current flowing through the inductor and $u_L(t)$ the voltage drop across the inductor.

\[ i(t) \quad \rightarrow \quad U_L(t) \quad \text{Symbol of an inductor} \]

- $i_C(t) = C \frac{du_C(t)}{dt}$, where $i(t)$ is the current flowing through the capacitor and $u_C(t)$ the voltage drop across the capacitor.

\[ i(t) \quad \rightarrow \quad U_C(t) \quad \text{Symbol of a capacitor} \]

C. Frequency domain

Under the hypothesis that the current and voltage are time dependent following a sine law, the following relations can be written using the complex notation.

- For the resistor $u_R = Ri$, where $u_R$ is the voltage drop across the resistor and $i$ the current flowing through it,
- For the inductance $u_L = j\omega Li$, where $u_L$ is the voltage drop across the inductor and $i$ the current flowing through it,
- For the capacitor $i_C = u_C j\omega C$, or, $u_C = \frac{i}{j\omega C}$, where $u_C$ is the voltage drop across the capacitance, and $i_C$ the current flowing through it.
The notion of impedance: definition

In the harmonic domain, the behavior of two terminal networks depends on frequency. The impedance of an electrical two terminal network in the harmonic domain is:

\[ Z_c = \frac{\mu}{i}, \]

where \( i \) is the current flowing through the network, and \( \mu \) the voltage drop across it. The impedance therefore measures the reaction of a network to the current \( i \) flowing through it.

The impedance is a complex number:
- Its magnitude is the ratio of the current and voltage magnitudes.
- Its phase is a measure of the delay between the voltage and current at a certain frequency.

Complément : Additional resources

Animation: ³

http://www.animations.physics.unsw.edu.au⁴

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³ - http://www.pedagogie.ac-nantes.fr/
Notions of impedance

B. Admittance versus Impedance

The admittance is defined as the inverse of the impedance. The electrical admittance $Y_e$ is written

\[ Y_e = \frac{1}{Z_e}. \]

**Méthode : Use of the admittance**

In the case of two dual terminal networks connected in parallel, the equivalent admittance $Y_{eq}$ is the sum of the admittances $Y_1$ and $Y_2$ of each network:

\[ Y_{eq} = Y_1 + Y_2. \]

**Méthode : Use of the impedance**

In the case of two dual terminal network connected in series, the equivalent impedance $Z_{eq}$ is the sum of the impedances $Z_1$ and $Z_2$ of each of the networks:

\[ Z_{eq} = Z_1 + Z_2. \]

C. Definitions

In the following figure, we note:

- the source impedance: The internal resistance of the generator (voltage or current) which is considered ideal
- the load impedance: The impedance presented by the load.

The internal impedance of the generator can be measured at its output when turned off.

D. Interaction generator-receiver

When a generator is connected to a load, there are two way of proceeding.
Optimising the voltage transfer

\[ \frac{U_s}{U_0} \text{ should be maximum. This ratio can be written: } \frac{U_s}{U_0} = \frac{Z_C}{Z_C + Z_i}. \text{ In this case } Z_C >> Z_i, \text{ so that } U_s \approx U_0. \]

Optimising the power transfer

For the maximum power to be transmitted from the source to the load, the calculations (detailed here, here, and here) show that the relation 
\[ Z_i = Z_C^* \]
where 
\[ Z_C^* \text{ is the complex conjugate of } Z_C. \]

Complément : Additional resources

A transformer can be used to modify the apparent internal impedances of the generator and load to maximise the power transfer.

5 - http://uel.unisciel.fr/physique/sinusoi/sinusoi_ch04/co/apprendre_ch4_06.html
6 - http://en.wikipedia.org/wiki/Impedance_matching
7 - http://en.wikipedia.org/wiki/Maximum_power_transfer_theorem
A. Coupling equations

Shown below is a virtual lossless transformer. It is used to simulate real electric transformers for coupling effects in electroacoustics.

![Ideal transformer diagram]

The equations which describe the behavior of an ideal transformer are:

\[ n_1 i_1 = n_2 i_2 \]  \hspace{1cm} (5)

\[ \frac{u_1}{n_1} = \frac{u_2}{n_2} \]  \hspace{1cm} (6)

where \( n_1 \) and \( n_2 \) represent the number of primary (subscript 1 on schematic) and secondary (subscript 2 on schematic) windings.

**Attention : Caution**

The electrical impedance:

\[ Z_1 = \frac{u_1}{i_1} \]

at the input of a transformer depends on the impedance at the output:

\[ Z_2 = \frac{u_2}{i_2} \]
An illustration of the ideal transformer is given in this lecture\(^8\) and here\(^9\).

### B. The gyrator

An ideal gyrator is a two port network whose input (respectively output) voltage (respectively output) is directly proportional to the output (respectively input) current. The ratio \(\alpha\) is usually called the "gyration resistance". In the case of this lecture, we will use the "coupling factor" (for reasons that will become apparent in the following lectures, see section 3.2\(^{10}\)).

In the case of an asymmetrical representation, a gyrator is illustrated by:

![Gyrator Diagram](https://example.com/gyrator_diagram.png)

The relations between current and voltage are written:

\[
\begin{align*}
u_1 &= \alpha i_2 \\
\nu_2 &= \alpha i_1 \\
\end{align*}
\]

where \(n_1\) and \(n_2\) represent the number of primary (subscript 1 on schematic) and secondary (subscript 2 on schematic) windings.

---

**Attention : Caution**

The electrical impedance \(Z_1 = \frac{\nu_1}{i_1}\) at the input of the gyrator depends on the admittance at the output \(Y_2 = \frac{i_2}{\nu_2}; Z_1 = \alpha^2 Y_2\).

### C. To know more

Detailed lectures on quadripoles are given here:

- [http://ressources.univ-](http://ressources.univ-)
- [http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/transf.html](http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/transf.html)
- [../../Grain3.2en/index.html](../../Grain3.2en/index.html)
Common quadripoles

lemans.fr/AccesLibre/UM/Pedago/physique/02/cours_elec/quadripo.pdf


11 - http://ressources.univ-lemans.fr/AccesLibre/UM/Pedago/physique/02/cours_elec/quadripo.pdf
Electrical circuit analysis

A. Kirchoff's Law

In a circuit, it is possible to calculate the voltage drops on each dipole, and the current intensity in each branch of the circuit by applying the two Kirchoff laws: the node law and the loop law.

Complément : Additional resources
More information

Millman's theory is a particular form of the node law in which the currents are described by the voltage drops.

Complément : Additional resources
More information

14 - http://subaru.univ-lemans.fr/AccesLibre/UM/Pedago/physique/02/electri/kirchhoff.html
15 - http://ressources.univ-lemans.fr/AccesLibre/UM/Pedago/physique/02/cours_elec/millman.html
B. Voltage divider

The voltage divider is a simple electrical circuit that divides the input voltage by a certain value. This value is determined by the ratio between two resistors. The input voltage is applied across the two resistors and the output is taken between the two resistors. The relationship between the output and input is:

\[ U_2 = U_0 \frac{R_2}{R_1 + R_2} \]

Complément : Additional resources

More information on the voltage divider here\(^\text{16}\)

C. Current divider

The current divider is an electric circuit that allows the division of the input current by a ratio of resistor values.

A circuit composed of two resistors in parallel can be used to this end. The input current is applied to the circuit, and the output current flows through one of the resistors.

\[ i_2 = i_0 \frac{R_1}{R_1 + R_2} \]
D. Thevenin generator

The notion of equivalent Thevenin generator implies that we model a real generator with the perfect voltage source $U_0$ connected in series with a resistor $Z_{\text{internal}}$ which represents the internal impedance of the generator. This impedance will influence the output voltage $U_s$, which will no longer be constant for any value of the load $Z_{\text{charge}}$.

![Thevenin generator diagram]

E. Norton generators

The notion of Norton generator allows us to represent an electrical source with an ideal current generator $I_0$ connected in parallel with an internal admittance $Y_{\text{internal}}$. This admittance means the output current $I_s$ will not be constant for every load value $Y_{\text{charge}}$.

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18 - http://subaru.univ-lemans.fr/AccesLibre/UM/Pedago/physique/02/electri/thevenin.html
**Complément : Additional resources**

More information on Norton generators can be found [here](http://subaru.univ-lemans.fr/AccesLibre/UM/Pedago/physique/02/electri/norton.html)
A. Summary

- The physical quantities introduced here are the voltage $u$ and the current $i$.
- The relations between the basic electrical elements are the following (using the complex notation):
  - For a resistor: the voltage and current are proportional $u = Ri$.
  - For an inductance: the voltage follows the time derivative of the current $u = j\omega Li$.
  - For a capacitance: the voltage follows the time integral of the current $u = \frac{i}{j\omega C}$.
- The impedance of an electrical system describes the reaction of a system (voltage drop across the system) for a given current that flows through it. The impedance is a complex (magnitude, phase), frequency dependent number.
- The admittance is defined as the inverse of the impedance.
- The most common electroacoustic quadripoles are the ideal transformer and gyrator.

B. Test your knowledge

Exercice 1 : Test your knowledge

Question 1

For an inductance $L$, the relation between the voltage $u$ and the current $i$ is
Question 2

The impedance of a two terminal network

- Is always a real number
- Can be a pure imaginary number
- Is the ratio of current over voltage
- Is the ratio of voltage over current
- Does not depend on frequency

Question 3

Association of two resistances of the same value. For two resistors $R$ in series, express the value of the total impedance

- $R/2$
- $R$
- $2R$

Question 4

Association of two resistances of the same value. For two resistors $R$ in parallel, express the value of the total impedance

- $R/2$
- $R$
- $2R$

Question 5

Association of two impedances of the same value. For two inductors $L$ in series,
express the value of the total impedance

- $L/2$
- $L$
- $2L$. 

**Question 6**

Association of two impedances of the same value. For two inductors $L$ in parallel, express the value of the total impedance

- $L/2$
- $L$
- $2L$. 

**Question 7**

Association of two impedances of the same value. For two capacitors $C$ in series, express the value of the total impedance

- $C/2$
- $C$
- $2C$. 

**Question 8**

Association of two impedances of the same value. For two capacitors $C$ in parallel, express the value of the total impedance

- $C/2$
- $C$
- $2C$. 

Conclusion
C. Exercise 1: Series RLC circuits

Question

Write the impedance \( Z = \frac{u}{i} \) of the following system comprised of the elements \( R, L \) and \( C \) connected in series.

\[
\begin{align*}
&\text{Series RLC circuit} \\
&i \quad R \quad L \quad C \quad u
\end{align*}
\]

D. Exercise 2: Parallel RLC circuit

Question

Write the expression of the admittance \( Y = \frac{i}{u} \) of the system comprised of the elements \( R, L \) and \( C \) connected in parallel. Deduce the expression of the impedance.

\[
\begin{align*}
&\text{Parallel RLC circuit} \\
&i \quad R \quad L \quad C \quad u
\end{align*}
\]
Bibliography

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- Two port network, \textit{Wikipedia}\textsuperscript{24}
- Pierre Muret, Systèmes linéaires à temps continu : quadripôles, filtrage et synthèse des filtres, Université Joseph Fourier, Grenoble\textsuperscript{25} (in French)

\textsuperscript{21} - http://uel.unisciel.fr/physique/sinusoi/sinusoi/co/sinusoi.html
\textsuperscript{22} - http://ressources.univ-lemans.fr/AccesLibre/UM/Pedago/physique/02/electri/rlcserie.html
\textsuperscript{23} - http://ressources.univ-lemans.fr/AccesLibre/UM/Pedago/physique/02/cours_elec/quadripo.pdf
\textsuperscript{24} - http://en.wikipedia.org/wiki/Two-port_network